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An Equation for the Field Amplitude in Geometrical Optics

IRVIN KAY

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Abstract

The geometrical optics approximation of solutions of the two dimensional wave equation for an isotropic inhomogeneous medium is considered. An ordinary differential equation for a quantity which is inversely proportional to the geometrical optics field intensity (the square of the field amplitude) is derived. This new equation along with the standard ray and phase equations form a system from which a complete wave solution in the optical limit can be calculated numerically, e.g., by means of an analog computer acting as a differential analyzer.

The examples of a homogeneous medium and a medium whose index of refraction is a function of one variable are discussed, and the results of the preceding analysis are verified in these two special cases.

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1. Introduction

The method of ray tracing or geometrical optics is a useful tool for the theoretical investigation of electromagnetic wave propagation in a general inhomogeneous medium. If the propagation takes place in the earth's atmosphere, for which the index of refraction varies quite slowly, the method is effective even at radio frequencies. By the use of analog computers in connection with ray tracing an analysis of a number of interesting phenomena observed experimentally in tropospheric propagation can be carried out on models for which a realistic variation of the index of refraction may be assumed.[1]

The direct information one can obtain by the use of the ray tracing technique is in the form of the ray paths and the approximate phase of a propagating wave. If these quantities can be obtained analytically, in the form of an explicit solution of the ray equations, one can also obtain the approximate field amplitude analytically.[2] If, however, numerical integration methods are used to obtain the ray paths and phase the field amplitudes are less easy to find. A popular way of estimating the relative variation of field amplitude in space is to observe from a graphical representation the spatial variation of the density of the rays. In addition to being rather inelegant this method seems likely to be less accurate than is necessary since there does exist an analytical connection between the approximate field amplitude, the ray paths and the approximate phase.

In addition, in a region where the rays become extraordinarily dense, such as the neighborhood of a caustic, or where the rays disappear entirely in skip zones or shadow regions the graphical procedure for obtaining the approximate field amplitude breaks down.

In view of these difficulties it seems desirable, therefore, to investigate the possibility of extending the geometrical optics method for treating propagation in an inhomogeneous medium, especially in connection with numerical techniques and the use of an analog computer. This article will deal with the problem of obtaining the approximate field amplitude in the case where the field of rays is regular. An ordinary differential equation which is coupled to the ray equations and whose solution is inversely proportional to the square of the field amplitude in the geometrical optics limit will be derived. This differential equation can be integrated numerically by means of the same procedures which are applied to the ray equations. It may also prove to be of some theoretical value.

A future article will deal with the problem of extending the geometrical optics theory to caustics and shadow regions.

This article will be concerned only with the scalar wave equation in two dimensions. In the optical limit polarization effects are usually negligible, and for practical applications the two dimensional problem is often sufficient. Considerable difficulties arise if an attempt is made to apply the analysis given here to the three dimensional problem.

For the sake of completeness Section 2 of this article will deal with properties of the geometrical optics solution of the wave equation that are more or less well known. In Section 3 the differential equation whose solution is inversely proportional to the square of the geometrical optics field amplitude is derived. In Section 4 two examples are given to illustrate the validity of the differential equation.

2. The Geometrical Optics Solution of the Wave Equation.

The wave equation in two dimensions for an inhomogeneous medium whose optical properties are given by the index of refraction $n(x,y)$ has the form

$$u_{xx} + u_{yy} + k^2 n^2 u = 0 , \tag{1}$$

where k is the wave number $\frac{\omega}{c}$ with ω the angular frequency and c the velocity of light in a vacuum. It will be assumed that $n(x,y)$ is positive everywhere.

The standard formal procedure for obtaining the optical approximation for u in (1) is to assume a form

$$u(x,y) = w(x,y) e^{ik\psi(x,y)} \tag{2}$$

and to substitute (2) into (1). The result is

$$k^2 \left\{ n^2 - (\nabla\psi)^2 \right\} w + ik \left\{ 2\nabla\psi \cdot \nabla w + (\Delta\psi) w \right\} + \Delta w = 0 . \tag{3}$$

The function ψ , called the eiconal, is now chosen so that it satisfies the eiconal equation

$$(\nabla\psi)^2 = n^2 . \tag{4}$$

The curves ψ = constant are the geometrical optics wavefronts, the orthogonal trajectories of which are the rays of the optical system. Next, it is assumed that there is a function $A(x,y)$ such that for x and y fixed

$$\lim_{k \to \infty} w = A \quad \text{and} \quad \lim_{k \to \infty} \nabla w = \nabla A . \tag{5}$$

It is also assumed that

$$\lim_{k \to \infty} \frac{\Delta w}{k} = 0 . \tag{6}$$

After the application of (4), (5) and (6) to (3) there results

$$2\nabla\psi\cdot\nabla A + (\Delta\psi)A = 0 , \tag{7}$$

which is the transport equation for the field amplitude A in the optical limit.

The first order partial differential equation (4) for the eiconal is equivalent to a system of ordinary differential equations for the rays of the optical system according to the theory of Hamilton and Jacobi.[3] The ray equations are

$$\begin{aligned} \dot{x} &= p , \quad \dot{p} = \tfrac{1}{2}(n^2)_x \\ \dot{y} &= q , \quad \dot{q} = \tfrac{1}{2}(n^2)_y \\ p^2 &+ q^2 = n^2 , \end{aligned} \tag{8}$$

where the dots indicate differentiation with respect to a parameter τ in terms of which the rays are described:

$$\begin{aligned} x &= x(\tau) \\ y &= y(\tau) . \end{aligned}$$

The last equation of (8) is an integral of the first four equations,* so that it holds for all values of τ if the initial values of the system at some value τ_o of τ are selected in such a way that it holds initially:

$$p^2(\tau_o) + q^2(\tau_o) = n^2\left[x(\tau_o), y(\tau_o)\right] .$$

---

* For $\frac{d(n^2)}{d\tau} = (n^2)_x\,\dot{x} + (n^2)_y\,\dot{y} = 2(p\dot{p} + q\dot{q}) = \frac{d}{d\tau}(p^2 + q^2)$.

The eiconal $\psi(x,y)$ is thus given by

$$\psi(x,y) = \int_{\tau_0}^{\tau} n^2[x(\tau),y(\tau)]d\tau + \psi[x(\tau_0),y(\tau_0)] \ , \tag{9}$$

a fact which can be demonstrated by showing that

$$p = \psi_x, \quad q = \psi_y \ . \tag{10}$$

It is convenient for the purposes of this article to use the eiconal ψ, itself, rather than τ as the ray parameter. To do this let us consider the system generated by the prescription of initial values on some initial curve

$$\psi(x,y) = \psi_0 = \text{constant} \ , \tag{11}$$

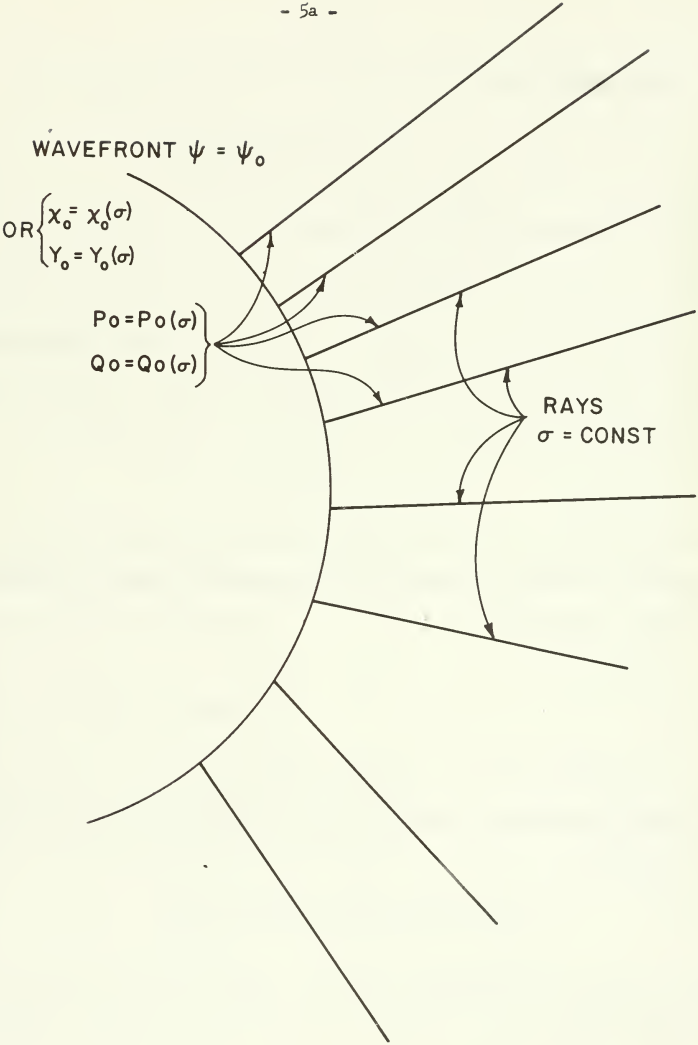
which is an initial value of the eiconal, and describes the initial position of a wavefront. We can suppose that the curve described by (11) can be expressed in terms of a parameter σ:

$$\begin{array}{lcll} x(\psi_0,\sigma) = x_0(\sigma) & & p(\psi_0,\sigma) = p_0(\sigma) & \\ & \text{with} & & \text{and } p_0^2(\sigma)+q_0^2(\sigma)=n^2(x_0,y_0) \ . \\ y(\psi_0,\sigma) = y_0(\sigma) & & q(\psi_0,\sigma) = q_0(\sigma) & \end{array} \tag{12}$$

The ray solution of (8) will be a pair of functions $x(\psi,\sigma)$, $y(\psi,\sigma)$ having the property that

$$\dot{x} = x_\psi \dot{\psi}, \quad \dot{y} = y_\psi \dot{\psi}, \quad \dot{p} = p_\psi \dot{\psi}, \quad \dot{q} = q_\psi \dot{\psi} \tag{13}$$

since τ and σ will be independent variables. From (9) and (13) the equations (8) can be written


WAVEFRONT $\psi = \psi_0$
OR $\begin{cases} X_0 = X_0(\sigma) \\ Y_0 = Y_0(\sigma) \end{cases}$
$\left. \begin{array}{l} Po = Po(\sigma) \\ Qo = Qo(\sigma) \end{array} \right\}$
RAYS
σ = CONST

INITIAL WAVEFRONT SURFACE $\psi = \psi_o$

$$x_\psi = \frac{p}{n^2}, \quad p_\psi = \frac{(n^2)_x}{2n^2}$$

(14)
$$y_\psi = \frac{q}{n^2}, \quad q_\psi = \frac{(n^2)_y}{2n^2}$$

$$p^2 + q^2 = n^2 .$$

It can be demonstrated easily that with the initial conditions (12) and if the first four equations of (14) are satisfied, the last equation of (14) holds automatically. For from the first four equations of (14) it follows that

$$\frac{\partial(n^2)}{\partial\psi} = \frac{\partial(n^2)}{\partial x} x_\psi + \frac{\partial(n^2)}{\partial y} y_\psi = 2pp_\psi + 2qq_\psi = \frac{\partial}{\partial\psi}(p^2 + q^2) . \tag{15}$$

After an integration of (15) with respect to ψ there results

$$p^2 + q^2 = n^2 + f(\sigma) , \tag{16}$$

where $f(\sigma)$ is a function of σ alone. But if ψ is set equal to ψ_0 in (16), then the last equation of (14) implies that $f(\sigma) = 0$, and the demonstration is completed.

It can be demonstrated similarly with the aid of (14) that the vector (p,q) is orthogonal to the vector (x_σ, y_σ) everywhere if this is true initially. Consider

(17)
$$\begin{aligned} \frac{\partial}{\partial\psi}(px_\sigma + qy_\sigma) &= p_\psi x_\sigma + q_\psi y_\sigma + px_{\psi\sigma} + qy_{\psi\sigma} \\ &= \frac{(n^2)_x}{2n^2} x_\sigma + \frac{(n^2)_y}{2n^2} y_\sigma + p \frac{\partial}{\partial\sigma}\left(\frac{p}{n^2}\right) + q \frac{\partial}{\partial\sigma}\left(\frac{q}{n^2}\right) \\ &= \frac{1}{2n^2}\frac{\partial(n^2)}{\partial\sigma} + \frac{1}{2n^2}\frac{\partial(n^2)}{\partial\sigma} - \frac{1}{n^2}\frac{\partial(n^2)}{\partial\sigma} = 0 , \end{aligned}$$

where again (14) has been used. It follows from (17) that

$$px_\sigma + qy_\sigma = g(\sigma) \,, \tag{18}$$

where $g(\sigma)$ is a function of o alone. If the initial conditions in (12) are chosen so that

$$p_o(\sigma)x_o'(\sigma) + q_o(\sigma)y_o'(\sigma) = 0 \,, \tag{19}$$

where the primes indicate differentiation with respect to σ, then by setting $\psi = \psi_o$ in (19) it follows that $g(\sigma) \equiv 0$; this completes the demonstration. This orthogonality condition as well as the last equation of (12) can be satisfied initially by the choice

$$\begin{aligned} p(\psi_o,\sigma) &= p_o(\sigma) = \mp n(x_o,y_o)y_o'(\sigma)\left\{x_o'(\sigma)^2 + y_o'(\sigma)^2\right\}^{-\frac{1}{2}} \\ q(\psi_o,\sigma) &= q_o(\sigma) = \pm n(x_o,y_o)x_o'(\sigma)\left\{x_o'(\sigma)^2 + y_o'(\sigma)^2\right\}^{-\frac{1}{2}} \,. \end{aligned} \tag{20}$$

Now the vector (ψ_x,ψ_y) is clearly orthogonal to the vector (x_σ,y_σ) everywhere, for

$$0 = \frac{\partial\psi}{\partial\sigma} = \psi_x\, x_\sigma + \psi_y\, y_\sigma \,. \tag{21}$$

Hence the vector (p,q) is proportional to the vector (ψ_x,ψ_y). Moreover, from (14) there results

$$\frac{p\psi_x + q\psi_y}{n^2} = \psi_x\, x_\psi + \psi_y\, y_\psi = 1 \,. \tag{22}$$

From (22) and the last equation of (14) it follows that the proportionality factor between (p,q) and (ψ_x,ψ_y) must be the constant 1. Thus the relations (10) hold everywhere, and because of the last equation in (8) it follows that the eiconal equation (4) is satisfied everywhere.

The system of differential equations (14) along with the initial conditions (12) and (20) determines a coordinate transformation

$$x = x(\psi,\sigma)\ ,\quad y = y(\psi,\sigma)\ . \tag{23}$$

The transformation (23) is a one-to-one mapping of any region Γ in the (ψ,σ) plane into a region Γ' in the (x,y) plane provided the Jacobean determinant

$$D = \frac{\partial(x,y)}{\partial(\psi,\sigma)} = x_\psi y_\sigma - x_\sigma y_\psi \tag{24}$$

is different from zero and exists throughout Γ. The (ψ,σ) coordinate system is orthogonal as one can see easily from (21), (14) and (10). The rays σ = constant and the wavefronts ψ = constant of the geometrical optics system form the coordinate lines. The coordinate system is singular where the determinant D of (24) vanishes, and this happens generally on a curve which is the envelope of rays, that is, a caustic of the ray system.*

We now consider the geometrical optics amplitude A of the field. Since by (4) the magnitude of $\nabla\psi$ is n we can write (7) as

$$2n^2 \frac{\partial A}{\partial \psi} + (\Delta\psi)A = 0\ , \tag{25}$$

for the arclength differential ds along a ray is given by

$$ds = (x_\psi{}^2 + y_\psi{}^2)^{\frac{1}{2}}\, d\psi = \frac{1}{n}\, d\psi\ ,$$

---

*It may also happen on a focus which is a single point where the rays intersect.

according to (14), and the dot product of a unit vector in the ray direction with ∇A is $\frac{dA}{ds}$.

If we define $(\nabla\sigma)^2 = \nu^2$ and take into account (4) we can express the laplacian Δ in the (ψ,σ) coordinates:

$$\Delta = n^2 \frac{\partial^2}{\partial\psi^2} + \nu^2 \frac{\partial^2}{\partial\sigma^2} + \frac{1}{D} \frac{\partial(n^2 D)}{\partial\psi} \frac{\partial}{\partial\psi} + \frac{1}{D} \frac{\partial(\nu^2 D)}{\partial\sigma} \frac{\partial}{\partial\sigma},$$

where D is given by (24).[4] Thus we have

$$(26) \qquad \Delta\psi = \frac{1}{D} \frac{\partial(n^2 D)}{\partial\psi} .$$

After a substitution from (26) into (25) there results

$$(27) \qquad \frac{\partial A}{\partial\psi} + \frac{1}{2} \left\{ \frac{\partial(n^2 D)}{\partial\psi} /(n^2 D) \right\} A = 0 .$$

The solution of (27) is

$$(28) \qquad A = A_0(\sigma)/(nD^{1/2}) ,$$

where $A_0(\sigma)$ is a function of σ alone and is determined by an initial condition for $A(\psi,\sigma)$ on each ray σ = constant. It is clear from (28) that A becomes singular on a caustic where D vanishes.

3. The Field Amplitude.

It is now possible to add to the differential equations (14) equations whose solution will provide the quantity D of (24) as a function of ψ and σ, so that by making use of (28) one can find the field amplitude A along any ray.

If the rays $x(\psi,\sigma)$, $y(\psi,\sigma)$ were known analytically it would be possible to determine D directly from its definition (24). If numerical integration methods are to be used, however, as matters now stand D can only be estimated in this case by means of a graphical representation of the rays and an appropriate geometrical interpretation of D: for example, in its role as the ratio of crossections of two different wavefronts cut out by an infinitesimal tube of rays.[2]

Consider a quantity $M(\psi,\sigma)$ defined by

$$M = n^2 D = p y_\sigma - q x_\sigma , \tag{29}$$

where (24) and (14) have been used. According to (28) $A(\psi,\sigma)$ is inversely proportional to the square root of M, with a proportionality factor which is independent of ψ.

With the help of (14) and (29) the derivative M_ψ can be expressed as

$$M_\psi = \frac{1}{2n^2} \frac{\partial(n^2)}{\partial x} y_\sigma + p\, y_{\sigma\psi} - \frac{1}{2n^2} \frac{\partial(n^2)}{\partial y} x_\sigma - q\, x_{\sigma\psi} . \tag{30}$$

By the use of (29) and the orthogonality of the vectors (p,q) and (x_σ, y_σ) [cf. (18) and the discussion following it] x_σ and y_σ can be expressed in terms of p,q and M. That is, the solution of the linear equations

$$\begin{aligned} p y_\sigma - q x_\sigma &= M \\ q y_\sigma + p x_\sigma &= 0 \end{aligned} \tag{31}$$

is

$$\begin{aligned} x_\sigma &= - qM/n^2 \\ y_\sigma &= pM/n^2 \end{aligned} \tag{32}$$

After the substitution of (32) into (30) with the help of (14) there results

$$M_\psi = \left\{ \frac{1}{2n^4} \frac{\partial(n^2)}{\partial x} p + \frac{1}{2n^4} \frac{\partial(n^2)}{\partial y} q \right\} M + p \frac{\partial}{\partial\sigma} (q/n^2) - q \frac{\partial}{\partial\sigma} (p/n^2) .$$

The derivatives in the last two terms on the right of this equation can be expanded by the use of (32) and the last equation of (14):

$$M_\psi = \frac{1}{2n^4} \left\{ q + p^2/q \right\} \frac{\partial(n^2)}{\partial y} M + \frac{p}{2n^2 q} \frac{\partial(n^2)}{\partial\sigma} - \frac{p^2 p_\sigma}{n^2 q} - \frac{q p_\sigma}{n^2} ,$$

whence from the last equation of (14) there results

$$M_\psi = \left\{ \frac{\partial(n^2)}{\partial y} /(2n^2 q) \right\} M - p_\sigma/q . \tag{33}$$

After differentiating the second equation in (14) with respect to σ and making use of (32) we obtain

$$p_{\sigma\psi} = \frac{1}{2n^2} \left\{ p \frac{\partial}{\partial y} \left[\frac{\partial(n^2)}{\partial x} /n^2 \right] - q \frac{\partial}{\partial x} \left[\frac{\partial(n^2)}{\partial x} /n^2 \right] \right\} M . \tag{34}$$

Equations (33) and (34) can be appended to the system (14), thus enlarging the system of ray equations so that a complete solution will give p_σ and M as well as x, y, p and q along a ray. Initial conditions at $\psi = \psi_o$ for M and p_σ are

$$\begin{aligned} M_o(\sigma) &= p_o(\sigma) y_o'(\sigma) - q_o(\sigma) x_o'(\sigma) \\ p_{\sigma o} &= p_o'(\sigma) . \end{aligned} \tag{35}$$

By the elimination of p_σ the equations (33) and (34) can be reduced to a single second order linear differential equation for M. Thus, after solving (33) for p_σ and differentiating with respect to ψ we obtain

$$(36) \qquad p_{\sigma\psi} = \frac{1}{2n^2}\left\{p\,\frac{\partial}{\partial x}\left[\frac{\partial(n^2)}{\partial y}\Big/n^2\right] + q\,\frac{\partial}{\partial y}\left[\frac{\partial(n^2)}{\partial y}\Big/n^2\right]\right\}M - q\,M_{\psi\psi}\,.$$

From the substitution of (36) into (34) there results

$$(37) \qquad M_{\psi\psi} - \left\{\Delta(\log n^2)/(2n^2)\right\}M = 0\,.$$

The first of the relations in (35) provides one initial condition for M in (37). The other initial condition can be obtained from (33) and the second equation of (35). The initial conditions are, then:

$$(38) \qquad \begin{aligned} M_o(\sigma) &= p_o(\sigma)y_o'(\sigma) - q_o(\sigma)x_o'(\sigma) \\ M_{\psi o}(\sigma) &= \left\{\left[\frac{\partial(n^2)}{\partial y}\right]_o\Big/(2n_o^2 q_o)\right\}M_o - p_o'/q_o\,, \end{aligned}$$

where, as usual the subscript o attached to a quantity means the value of the quantity at $\psi = \psi_o$.

If n^2 is a constant, indicating that the medium is homogeneous, (37) reduces to

$$(39) \qquad M_{\psi\psi} = 0\,.$$

The same equation results if $\log n^2$ is a harmonic function. This fact is consistent with the well-known fact that when $\log n^2$ is a harmonic function there exists a coordinate transformation under which the wave equation (1) for an inhomogeneous medium reduces to a wave equation for a homogeneous medium, that is, with a constant index of refraction. The proof is simple; for the sake of completeness it will be given here.

Consider the coordinate transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) . \tag{40}$$

The wave equation (1) can be written

$$(\nabla\xi)^2\mu_{\xi\xi} + (\nabla\eta)^2\mu_{\eta\eta} + 2(\nabla\xi\cdot\nabla\eta)\mu_{\xi\eta} + (\Delta\xi)\mu_\xi + (\Delta\eta)\mu_\eta + k^2n^2\mu = 0 .$$

If

$$(\nabla\xi)^2 = (\nabla\eta)^2 = n^2, \quad \nabla\xi\cdot\nabla\eta = 0 \quad \text{and} \quad \Delta\xi = \Delta\eta = 0 , \tag{41}$$

the wave equation in (ξ,η) coordinates becomes

$$\mu_{\xi\xi} + \mu_{\eta\eta} + k^2\mu = 0 . \tag{42}$$

Now the conditions (41) can be satisfied if the function $w(z) = \xi + i\eta$, for $z = x + iy$, is an analytic function of z, i.e., if ξ and η are conjugate harmonic functions of x and y, and if

$$|w'(z)|^2 = n^2 . \tag{43}$$

The condition (43), on the other hand, implies that $\log n^2$ is the real part of the analytic function $\left\{w'(z)\right\}^2$ and hence that $\log n^2$ is harmonic. The steps in the proof given thus far are reversible; thus it follows that a sufficient condition for the existence of a coordinate transformation (40), under which the wave equation (1) for an inhomogeneous medium goes into the wave equation (42) for a homogeneous medium, is that $\log n^2$ be a harmonic function.

To explain the consistency of the property (39) of M in this case let M^{ξ} be the quantity M in (29) which gives the field amplitude, according to (28), for the wave equation (42), in the (ξ,η) coordinate system, and let M be the corresponding quantity for the wave equation (1) in the (x,y) coordinate system. Then by (24) and (29) we have

$$M^{\xi} = \frac{\partial(\xi,\eta)}{\partial(\psi,\sigma)} \tag{44}$$

and

$$M = n^2 \frac{\partial(x,y)}{\partial(\psi,\sigma)} . \tag{45}$$

Now the well-known property of Jacobean determinants in two successive transformations:

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(\psi,\sigma)} = \frac{\partial(\xi,\eta)}{\partial(\psi,\sigma)}$$

implies along with (44), (45), the Cauchy-Riemann equations

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x ,$$

which must be satisfied when ξ and η are conjugate harmonic functions, and the conditions $(\nabla\xi)^2 = (\nabla\eta)^2 = n^2$ that

$$M = M^{\xi} . \tag{46}$$

According to (46) the quantity M must, indeed, satisfy the differential equation (39) if M^{ξ} does, and M^{ξ} satisfies (39) because the medium to which it corresponds is homogeneous, with index of refraction one according to (42). The above discussion proves, incidentally, that the condition

$$\Delta \log n^2 = 0$$

is also a necessary condition that there exist a coordinate transformation of the type introduced which transforms the wave equation for an inhomogeneous medium into one for a homogeneous medium.

4. Examples.

A. The Homogeneous Medium

Suppose the medium is homogeneous so that (39) holds. Then M is a linear function of ψ. Thus, by (38) we have

$$(47) \qquad M(\psi,\sigma) = -\left\{p_o'(\sigma)/q_o(\sigma)\right\}(\psi - \psi_o) + M_o(\sigma) ,$$

where

$$(48) \qquad M_o(\sigma) = p_o(\sigma)y_o'(\sigma) - q_o(\sigma)x_o'(\sigma) .$$

According to the standard geometrical optics result the field amplitude on a wavefront is proportional to the square root of the curvature of the wavefront.[2] By comparison with (28) and (29) this has the implication that M is proportional to the radius of curvature ρ of the wavefront. It is possible to verify that (47) agrees with the standard result, then, by expressing ρ as a function of ψ and σ.

Let t be the arclength parameter along a ray σ = constant. Then

$$(49) \qquad M = \frac{dM}{dt} / \frac{d\psi}{dt} = \frac{dM}{dt} / (\psi_x \frac{dx}{dt} + \psi_y \frac{dy}{dt}) = \frac{dM}{dt} /_n = n \frac{dD}{dt} ,$$

where the eiconal equation (4) and the definition (29) of M have been used along with the fact that $\nabla\psi$ has the direction of a ray and the fact that n is constant.

The unit vector ν along a ray normal to a wavefront, is given by

$$(50)\qquad \nu = (p,q)/n\ .$$

If we differentiate ν with respect to the arclength parameter s along the wavefront, we obtain

$$(51)\qquad \frac{d\nu}{ds} = \frac{1}{n}\frac{d\sigma}{ds}\frac{\partial p}{\partial\sigma}(1,\ -p/q) = \frac{1}{n}\frac{\partial p}{\partial\sigma}\left\{\left(\frac{\partial x}{\partial\sigma}\right)^2 + \left(\frac{\partial y}{\partial\sigma}\right)^2\right\}^{-\frac{1}{2}}(1,\ -p/q)\ .$$

A result of differential geometry is: the magnitude of $\frac{d\nu}{ds}$ is the curvature, or the reciprocal of the radius of curvature, of the wavefront. Thus, after making use of (32) and the last equation of (8), we obtain for the radius of curvature ρ of the wavefront

$$(52)\qquad \rho = \left|(Mq)/(n\frac{\partial p}{\partial\sigma})\right| = \left|(nDq)/\frac{\partial p}{\partial\sigma}\right|\ ,$$

where the definition (29) has also been applied. Since ρ is the distance from the point of tangency of a ray on a caustic to the point of intersection of the wavefront with the ray, ρ differs from the arclength t by a constant, and thus $d\rho = dt$. On the other hand, since n is constant, from (14) it follows that p and q are independent of ψ, i.e., functions of σ alone. These facts and (49) and (52) imply that

$$(53)\qquad M_\psi = \pm\, p_o'/q_o\ .$$

The sign in (53) can be determined by observing that $\frac{d\nu}{ds}$ has the direction of the vector (x_σ, y_σ), but according to (32) the vector $M(q, -p)$ has the opposite direction. From (51) and (52) it follows then that (53) must be

$$M_\psi = -p\sigma' / q_o \; ,$$

which verifies the result (47).

B. Ｔhe Medium Whose Index of Refraction is a Function of One Variable.

Suppose that n^2 is a function of x alone and that a plane wave is incident from $-\infty$; i.e., n^2 will be required to approach a constant. n_o^2 and the solution μ of (1) approaches $\exp\left\{ikn_o(x \cos \alpha + y \sin \alpha)\right\}$ as x approaches $-\infty$. In addition, it will be assumed that n^2 is smooth and everywhere positive and that $\lim\limits_{x \to \infty} \frac{d(n^2)}{dx} = 0$. The initial curve will be taken to be

$$y = \sigma, \quad x = -\infty. \tag{54}$$

The quantities p and q will be constant initially:

$$p(-\infty) = n_o \cos \alpha, \quad q(-\infty) = n_o \sin \alpha \; . \tag{55}$$

From (14) it follows that q is independent of ψ and thus will have the same value everywhere. It then follows from the last equation of (14) that

$$p = \sqrt{n^2 - n_o^2 \sin^2 \alpha} \; . \tag{56}$$

Equation (37) can be expressed in terms of the variables x and y. If it is assumed that M is independent of y, the resulting equation will be considerably simpler than it would be without this assumption. The final result will justify this assumption because of the uniqueness of the solution of (37), (38). From the first equation in (14) and equation (56) there results

$$(57) \qquad M_\psi = x_\psi M_x = pM_x/n^2 = \left\{n^2 - n_o^2 \sin^2\alpha\right\}^{\frac{1}{2}} Mx/n^2.$$

After using (57) in (37) we obtain

$$(58) \qquad (n^2 - n_o^2 \sin^2\alpha)(M'/n^2)' + M'(n^2)'/(2n^2) - \left\{(n^2)'/n^2\right\}' M = 0 ,$$

where the primes stand for differentiation with respect to x. Equation (58) can be rewritten as

$$(59) \qquad \left\{(n^2 - n_o^2 \sin^2\alpha)M'/n^2\right\}' - \left\{(n^2)'M/(2n^2)\right\}' = 0 .$$

After one integration (59) becomes

$$(60) \qquad (n^2 - n_o^2 \sin^2\alpha)M'/n^2 - (n^2)'M/(2n^2) = B ,$$

where B is a constant. The general solution of (60) is

$$(61) \qquad M = B(n^2 - n_o^2 \sin^2\alpha)^{\frac{1}{2}} \int_{-\infty}^{x} n^2(n^2 - n_o^2 \sin^2\alpha)^{-\frac{3}{2}} dx + C(n^2 - n_o^2 \sin^2\alpha)^{\frac{1}{2}} ,$$

sith B and C arbitrary constants. From the initial conditions (38), the initial surface given by (54) and equations (55) there results

$$(62) \qquad M(-\infty) = n_o \cos\alpha , \quad M_\psi(-\infty) = -\frac{d}{dy} \cot\alpha = 0 .$$

The first equation of (62) and equation (61) imply that

$$C = 1 . \tag{63}$$

The second equation of (62) and equation (61) imply since $\lim_{x \to -\infty} \frac{d(n^2)}{dx} = 0$, that

$$B = 0 . \tag{64}$$

Thus we have, finally,

$$M = (n^2 - n_o^2 \sin^2\alpha)^{\frac{1}{2}} . \tag{65}$$

Using (28) and (29) we obtain the field amplitude:

$$A = (n_o \cos \alpha)^{\frac{1}{2}}(n^2 - n_o^2 \sin^2\alpha)^{-\frac{1}{4}} , \tag{66}$$

where the first equation of (62) and the fact that the field amplitude approaches one as x approaches $-\infty$ have also been used. The result (66) agrees with that of Seckler and Keller[2] who calculated the geometrical optics solution of this problem using the standard method for obtaining the field amplitude, that is, computing the square root of the ratio of wavefront cross sections cut out by an infinitesimal tube of rays on the initial and final wavefronts.

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